Instructions. (30 points) Solve each of the following problems.

(1^{pts}) **1.** $\lim\limits_{x\to-4}\dfrac{x^2-16}{x-4}=$

(a) Does Not Exist

(b) 8

(✔) 0

(d) -8

Solution:

$$\begin{aligned}\lim_{x\to-4}\frac{x^2-16}{x-4}&=\frac{(-4)^2-16}{-4-4}\\&=\frac{16-16}{-8}\\&=\frac{0}{-8}=0\end{aligned}$$

(1^{pts}) **2.** If $\cos x=\dfrac{3}{5}$, $\dfrac{3\pi}{2}<x<2\pi$ then $\csc x=$

(a) $\dfrac{-3}{4}$

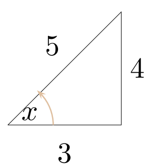
(✔) $\dfrac{-5}{4}$

(c) $\dfrac{-4}{3}$

(d) $\dfrac{-4}{5}$

Solution:

Since $\cos x=\dfrac{3}{5}=\dfrac{\text{adjacent}}{\text{hypotenuse}}$ we draw a triangle is shown below: Now, $H^2=A^2+O^2\Rightarrow 25=9+O^2\Rightarrow O=4$. Since $\dfrac{3\pi}{2}<x<2\pi$, then x lies in the forth quadrant. Hence since the $y-$ axis is negative, then $\sin x<0$ and since the $x-$axis is positive then $\cos x>0$. Therefore $\csc x<0$ Hence $\csc x=\dfrac{-5}{4}$.



(1^{pts}) **3.** $\lim\limits_{x\to\infty}\dfrac{3x^2+2}{1-x-x^3}=$

(✔) 0

(b) ∞

(c) $\dfrac{-1}{3}$

(d) -3

Solution:

Since $\lim\limits_{x\to\infty}(3x^3+2)=\infty$, $\lim\limits_{x\to\infty}(1-x-x^3)=\infty$ we have I.F. type ∞/∞. Divide each term in the numerator and each term in the denominator by the highest power in the denominator.

$$\lim_{x\to\infty}\frac{3x^2+2}{1-x-x^3}=\lim_{x\to\infty}\frac{\frac{3x^2+2}{x^3}}{\frac{1-x-x^3}{x^3}}$$
$$=\lim_{x\to\infty}\frac{\frac{3x^2}{x^3}+\frac{2}{x^3}}{\frac{1}{x^3}-\frac{x}{x^3}-\frac{x^3}{x^3}}$$
$$=\lim_{x\to\infty}\frac{\frac{3}{x}+\frac{2}{x^3}}{\frac{1}{x^3}-\frac{1}{x^2}-5}$$
$$=\frac{0+0}{0-0-1}=0$$

(1[pts]) **4.** $\cos\left(\frac{\pi}{18}\right)\cos\left(\frac{\pi}{9}\right)-\sin\left(\frac{\pi}{9}\right)\sin\left(\frac{\pi}{18}\right)=$

(a) 2 (b) ✔ $\frac{\sqrt{3}}{2}$

(c) 0 (d) $\frac{1}{2}$

Solution:

$$\cos\left(\frac{\pi}{18}\right)\cos\left(\frac{\pi}{9}\right)-\sin\left(\frac{\pi}{18}\right)\sin\left(\frac{\pi}{9}\right)=\cos\left(\frac{\pi}{18}+\frac{\pi}{9}\right)$$
$$=\cos\left(\frac{\pi}{6}\right)$$
$$=\frac{\sqrt{3}}{2}.$$

(1[pts]) **5.** The curve $f(x)=\dfrac{x^2-x-6}{x^3-4x}$ has a vertical asymptote at

(a) $y=0,\quad y=2$ (b) ✔ $x=0,\quad x=2$

(c) $x=0,\quad x=\pm2$ (d) $x=0,\quad x=-2$

Solution:

Write $f(x)=\dfrac{(x-3)(x+2)}{x(x-2)(x+2)}$. The zeroes of the denominator are -2, 0, and 2. To check that $x=-2$ is a vertical asymptote or not we take the limit at -2 from both sides. $\displaystyle\lim_{x\to-2}f(x)=\lim_{x\to-2}\frac{\cancel{(x+2)}(x-3)}{x\cancel{(x+2)}(x-2)}=\lim_{x\to-2}\frac{x-3}{x(x-2)}=\frac{-5}{8}$. Hence $x=-2$ is not a vertical asymptote. To check that $x=2$ we take the limit $\displaystyle\lim_{x\to2^+}f(x)=\lim_{x\to2^+}\frac{\overset{-}{(x-3)}(x+2)}{x\underset{+}{(x-2)}(x+2)}=-\infty$. Hence $x=2$ is a vertical asymptote. To check that $x=0$ we take the limit $\displaystyle\lim_{x\to0^+}f(x)=\lim_{x\to0^+}\frac{\overset{-}{(x-3)}(x+2)}{x\overset{-}{(x-2)}(x+2)}=\infty$. Hence $x=0$ is a vertical asymptote. Thus the function has vertical asymptote at $x=0$, and $x=2$.

(1$^{\text{pts}}$) **6.** If

$$f(x) = \begin{cases} x+3, & \text{if } x < -4; \\ 3, & \text{if } x = -4; \\ \dfrac{x^3+64}{x^2-16}, & \text{if } x > -4. \end{cases} \quad \text{Then } \lim_{x \to -4^+} f(x) =$$

(a) 0 (✔) -6

(c) 6 (d) -2

Solution:

Note that when computing limit as $x \to a^+$ means you approaches a from the right side that is $x > a$.

$$\begin{aligned}
\lim_{x \to -4^+} f(x) &= \lim_{x \to -4^+} \frac{x^3+64}{x^2-16} \\
&= \lim_{x \to -4} \frac{x^3+64}{x^2-16} && \text{A direct substation will give us I.F. } 0/0 \\
&= \lim_{x \to -4^+} \frac{(x+4)(x^2-4x+16)}{(x+4)(x-4)} && \text{Factoring } x+4 \text{ from denominator and numerator} \\
&= \lim_{x \to -4^+} \frac{\cancel{(x+4)}(x^2-4x+16)}{\cancel{(x+4)}(x-4)} \\
&= \lim_{x \to -4^+} \frac{x^2-4x+16}{x-4} \\
&= \frac{(-4)^2-4(-4)+16}{-4-4} \\
&= \frac{48}{-8} = -6.
\end{aligned}$$

(1$^{\text{pts}}$) **7.** $\dfrac{d^{52}}{dx^{52}}(\cos x) =$

(a) $-\sin x$ (b) $\sin x$

(✔) $\cos x$ (d) $-\cos x$

Solution:

Since $52 = 4(13) + 0$ then $\dfrac{d^{52}}{dx^{52}}(\cos x) = \dfrac{d^0}{dx^0}(\cos x) = \cos x$.

(1$^{\text{pts}}$) **8.** If $\dfrac{x^2-2x-3}{x-3} \le f(x) \le \sqrt{3x+7}$, $x \in [2,4], x \neq 3$, then $\lim_{x \to 3} f(x) =$

(a) 1 (✔) 4

(c) -4 (d) 0

Solution:

Since $\lim_{x \to 3} \dfrac{x^2-2x-3}{x-3} = \lim_{x \to 3} \dfrac{(x-3)(x+1)}{x-3} = \lim_{x \to 3}(x+1) = 4$, and $\lim_{x \to 3} \sqrt{3x+7} = 4$, then by The Sandwich Theorem we have $\lim_{x \to 3} f(x) = 4$.

(1^{pts}) **9.** $\lim\limits_{x\to 0} \dfrac{\sin^2(4x)}{x^2} =$

(✔) 16 (b) $\dfrac{1}{4}$

(c) $\dfrac{1}{16}$ (d) 4

Solution:

$$\begin{aligned}\lim_{x\to 0}\frac{x^2}{\sin^2(4x)} &= \lim_{x\to 0}\left(\frac{\sin(4x)}{x}\right)^2 \quad \tfrac{a^n}{b^n}=\left(\tfrac{a}{b}\right)^n.\\ &= \left(\lim_{x\to 0}\frac{4\sin(4x)}{4x}\right)^2 \text{make the top similar to the angle}\\ &= \left(4\lim_{x\to 0}\frac{\sin(4x)}{4x}\right)^2 \text{Use that } \lim_{x\to 0}\frac{\sin x}{x}=1=\lim_{x\to 0}\frac{x}{\sin x}\\ &= (4.1.)^2 = 16.\end{aligned}$$

(1^{pts}) **10.** $\lim\limits_{u\to 0} u\cos u\cot(3u) =$

(a) 3 (b) -3

(✔) $\dfrac{1}{3}$ (d) 1

Solution:

Direct substation will give us I.F. type 0/0.

$$\begin{aligned}\lim_{u\to 0} u\cos u\cot(3u) &= \lim_{u\to 0}\cos u\frac{u}{\tan(3u)}\\ &= \frac{1}{3}\lim_{u\to 0}\cos u\frac{3u}{\tan(3u)} \qquad \text{use } \lim_{\theta\to 0}\frac{\tan\theta}{\theta}=1\\ &= \frac{1}{3}(1)(1) = \frac{1}{3}\end{aligned}$$

(1^{pts}) **11.** If $y = \dfrac{\cot x}{1-\csc x}$, then $y' =$

(a) $\dfrac{\csc^2 x}{(1-\csc x)^2}$ (b) $\dfrac{-\csc x}{(1-\csc x)^2}$

(c) $\dfrac{-\csc x}{1-\csc x}$ (✔) $\dfrac{\csc x}{1-\csc x}$

Solution:

$$\begin{aligned}
y' &= \frac{(1-\csc x)(-\csc^2 x) - (\cot x)(\csc^x \cot x)}{(1-\csc x)^2} \\
&= \frac{-\csc^2 x + \csc^3 x - \csc x \cot^2 x}{(1-\csc x)^2} \\
&= \frac{\csc x(-\cot x + \cot^2 x - \csc^2 x)}{(1-\csc x)^2} && \text{Use the fact } \csc^2 x - \cot^2 x = 1 \\
&= \frac{\csc x(1-\csc x)}{(1-\csc x)^2} && \text{simplify} \\
&= \frac{\csc x}{1-\csc x}.
\end{aligned}$$

(1^{pts}) **12.** The graph of the function $F(x) = x^3 - 48x$, has a horizontal tangent line at

(✔) $x = \pm 4$ (b) $x = -4$

(c) $x = 0$ (d) $x = 4$

Solution:

$$\begin{aligned}
F(x) &= x^3 - 48x \\
F'(x) &= 3x^2 - 48 = 3(x^2 - 16) \\
F'(x) &= 0 \\
3(x^2 - 16) &= 0 \\
x &= \pm 4.
\end{aligned}$$

(1^{pts}) **13.** An equation for the tangent line to the curve $f(x) = x^2 - 2x$ at $x = -1$ is

(✔) $y = -4x - 1$ (b) $y = -4x + 7$

(c) $y = 4x + 1$ (d) $y = 4x - 1$

Solution:

The slope of the tangent line to $f(x) = x^2 - 2x$ at $x = -1$ is $f'(-1)$.

$$f(x) = x^2 - 2x \Rightarrow f'(x) = 2x - 2.$$

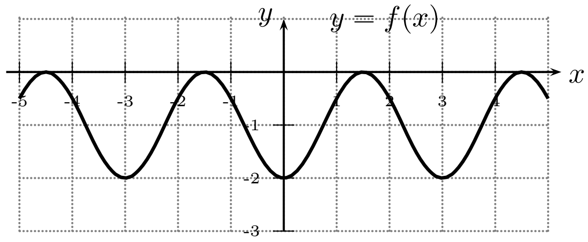
Hence

$$\text{the slope of the tangent } = f'(-1) = 2(-1) - 2 = -4.$$

Also $f(1) = (-1)^2 - 2(-1) = 3$. Now, we have $m = -4$ and $(-1, 3)$, hence

$$y - y_1 = m(x - x_1) \Rightarrow y - 3 = -4(x+1) \Rightarrow y - 3 = -4x - 4 \Rightarrow y = -4x - 1.$$

(1$^{\text{pts}}$) **14.** The accompanying figure shows the graph of $y = f(x)$. Then the period of $y = f(x)$ is



(✔a) 3

(b) π

(c) 1

(d) 2

Solution:
From the graph we can see that the function repeat itself every 3 units. Hence the period is 3

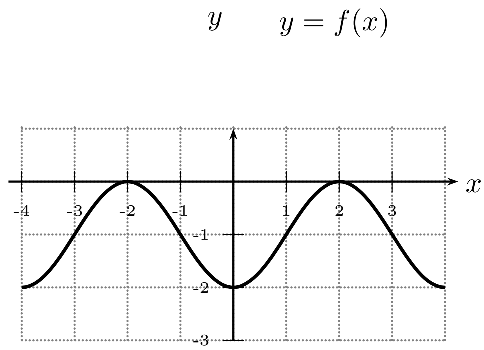
(1$^{\text{pts}}$) **15.** If $\lim_{x\to 1} f(x)$ exist and $f(1)$ is defined, then must $f(x)$ be continuous at $x = 1$.

(a) True

(✔b) False

Solution:
False. Let $f(x) = \begin{cases} 2, & \text{if } x \neq 1; \\ 0, & \text{if } x = 1. \end{cases}$ Then $f(1) = 0$, but $\lim_{x\to 1} f(x) = \lim_{x\to 1} 2 = 2$, Now, $\lim_{x\to 1} f(x) = 2 \neq 0 = f(1)$, hence f is discontinuous at $x = 1$.

(1$^{\text{pts}}$) **16.** The accompanying figure shows the graph of $y = f(x)$. Then $f'(-3) < f'(3)$.

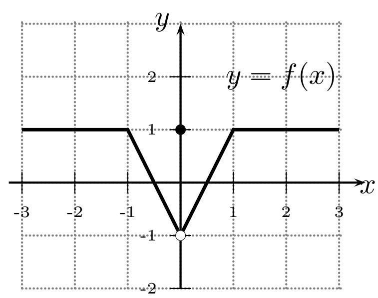


(a) True

(✔b) False

Solution:
From the graph we can see that the tangent line to the graph of $y = f(x)$ at -3 is rising up (left to right) and hence its slope is positive. Now, since the first derivative is the slope of the tangent line to the graph at -3 then $f'(-3) > 0$. From the graph we can see that the tangent line to the graph of $y = f(x)$ at 3 is falling down (left to right) and hence its slope is negative. Thus, since the first derivative is the slope of the tangent line to the graph at 3 then $f'(3) < 0$. Now, $f'(-3) > 0 > f'(3)$.

(1$^{\text{pts}}$) **17.** The accompanying figure shows the graph of $y = f(x)$. Then $\lim_{x\to 0} f(x) =$



(a) 0 (✔b) -1

(c) Does Not Exist (d) 1

Solution:

since $\lim_{x\to 0^+} f(x) = -1 = \lim_{x\to 0^-} f(x)$, then $\lim_{x\to 0} f(x) = -1$.

(1^{pts}) **18.** $\lim_{x\to 1} \dfrac{2-\sqrt{x+3}}{x-1} =$

(✔) $\dfrac{-1}{4}$ (b) $\dfrac{1}{4}$

(c) 0 (d) -4

Solution:

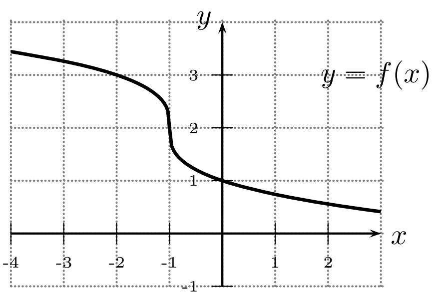
A direct substation will give us I.F. 0/0. Now,

$$\begin{aligned}
\lim_{x\to 1} \frac{2-\sqrt{x+3}}{x-1} &= \lim_{x\to 1} \frac{2-\sqrt{x+3}}{x-1} \cdot \frac{2+\sqrt{x+3}}{2+\sqrt{x+3}} && \text{Multiply by } 1 = \frac{2+\sqrt{x+3}}{2+\sqrt{x+3}} \\
&= \lim_{x\to 1} \frac{2^2-(\sqrt{x+3})^2}{(x-1)(2+\sqrt{x+3})} \\
&= \lim_{x\to 1} \frac{4-(x+3)}{(x-1)(2+\sqrt{x+3})} && \text{Simplify} \\
&= \lim_{x\to 1} \frac{-\cancel{(x-1)}}{\cancel{(x-1)}(2+\sqrt{x+3})} \\
&= \lim_{x\to -} \frac{-1}{2+\sqrt{x+3}} \\
&= \frac{-1}{\sqrt{1+3}+2} \\
&= \frac{-1}{4}.
\end{aligned}$$

(1^{pts}) **19.** The accompanying figure shows the graph of $y = f(x)$. Then f is differentiable at $x = -1$.

(a) True

(✔) False



Solution:

f is not differentiable at $x = -1$ since the derivative from the left of -1 approaches $-\infty$ and from the right of -1 approaches $-\infty$ and hence the function has a vertical tangent line at $x = -1$.

(1^{pts}) **20.** If $y = x + \cot x \tan x$, then $y' =$

(a) $-\sec^2 x \csc^2 x$ (b) 0

(✔) 1 (d) -1

Solution:

$$\begin{aligned} y &= x + \cot x \tan x \\ y &= x + \frac{1}{\tan x}\tan x \\ y &= x + 1 \\ y' &= 1. \end{aligned}$$

(1^{pts}) **21.** The function $f(x) = \dfrac{x+4}{3-|x+1|}$ is continuous on

(a) $(-\infty,-4)\cup(-4,\infty)$ (b) $(-\infty,-4)\cup(-4,2)\cup(2,\infty)$

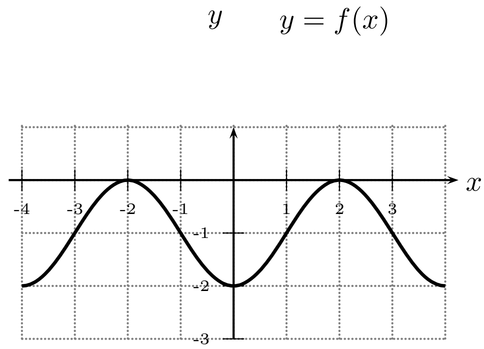
(c) $(-\infty,2)\cup(2,\infty)$ (d) $[-4,2]$

Solution: Notice that $f(x) = \dfrac{x+3}{3-|x+1|}$ is a rational function, then it is continuous on its domain which is the set of real number except the zeroes of the denominator and it is discontinuous at the zeroes of the the denominator. Now

$$\begin{aligned} 3-|x+1| = 0 &\Leftrightarrow |x+1| = 3 \\ &\Leftrightarrow x+1 = -3 \text{ or } x+1 = 3 \\ &\Leftrightarrow x = -4 \text{ or } x = 2. \end{aligned}$$

Hence f is continuous on $(-\infty,-4)\cup(-4,2)\cup(2,\infty)$.

(1^{pts}) **22.** The accompanying figure shows the graph of $y = f(x)$. Then $f'(2) =$



(a) -2

(b) 0

(c) Undefined

(d) 2

Solution:

From the graph we can see that the tangent line to the graph of $y = f(x)$ at 2 is horizontal ($y = 0$) and hence its slope is zero. Now, since the first derivative is the slope of the tangent line to the graph at 2 then $f'(2) = 0$.

(1^{pts}) **23.** If $y = x + \dfrac{1}{x^2} - \sqrt{5}$, then $y''' =$

(a) $\dfrac{-24}{x^5}$ (b) $\dfrac{-24}{x^5} + \dfrac{1}{2\sqrt{5}}$

(c) $\dfrac{24}{x^5} + \dfrac{1}{2\sqrt{5}}$ (d) $\dfrac{24}{x^5}$

Solution:

$$\begin{aligned} y &= x + \frac{1}{x^2} - \sqrt{5} \\ y &= x + x^{-2} - \sqrt{5} \\ y' &= 1 - 2x^{-3} \\ y'' &= 6x^{-4} \\ y''' &= -24x^{-5} = \frac{-24}{x^5}. \end{aligned}$$

(1^{pts}) **24.** If $y = \dfrac{3x-2}{x+2}$, then $y' =$

(a) $\dfrac{4}{(x+2)^2}$ (b) $\dfrac{6x+8}{(x+2)^2}$

(✔) $\dfrac{8}{(x+2)^2}$ (d) $\dfrac{-4}{(x+2)^2}$

Solution:

$$\begin{aligned} y' &= \frac{\overbrace{(x+2)}^{\text{Bottom}} \overbrace{(3x-2)'}^{\text{Derivative of Top}} - \overbrace{(3x-2)}^{\text{Top}} \overbrace{(x+2)'}^{\text{Derivative of Bottom}}}{\underbrace{(x+2)^2}_{}^{\text{Bottom}^2}} \\ &= \frac{(x+2)(3) - (3x-2)(1)}{(x+2)^2} \\ &= \frac{3x+6-(3x-2)}{(x+2)^2} \\ &= \frac{3x+6-3x+2)}{(x+2)^2} \\ &= \frac{8}{(x+2)^2}. \end{aligned}$$

(1^{pts}) **25.** $\displaystyle\lim_{\theta\to 0} \cos\left(\frac{\pi \tan(\theta)}{2\theta}\right) =$

(a) -1 (b) 1

(✔) 0 (d) ∞

Solution:

$$\begin{aligned} \lim_{\theta\to 0} \cos\left(\frac{\pi \tan(\theta)}{2\theta}\right) &= \cos\left(\lim_{\theta\to 0} \frac{\pi \tan(\theta)}{2\theta}\right) \\ &= \cos\left(\frac{\pi}{2}\right) \\ &= 0. \end{aligned}$$

(1^{pts}) **26.** If $y = x\cot x$, then $y' =$

(a) $\cot x - x\csc^2 x$ (b) $\cot x - x\csc x$

(c) $\cot x + x\csc x$ (d) $\cot x + x\csc^2 x$

Solution:

$$\begin{aligned} y' &= (x)'\cot x + x(\cot x)' \\ &= \cot x + x(-\csc^2 x) \\ &= \cot x - x\csc^2 x. \end{aligned}$$

(1^{pts}) **27.** $\lim\limits_{x\to 4} \dfrac{x^2-16}{x^2-x-12} =$

(a) $\dfrac{8}{7}$ (b) $\dfrac{7}{8}$

(c) 0 (d) $\dfrac{-8}{7}$

Solution:

$$\begin{aligned} \lim_{x\to 4} \frac{x^2-16}{x^2-x-12} &= \lim_{x\to 4} \frac{x^2-16}{x^2-x-12} && \text{A direct substation will give us I.F. } 0/0 \\ &= \lim_{x\to 4} \frac{(x+4)(x-4)}{(x-4)(x+3)} && \text{Factoring } x-4 \text{ from denominator and numerator} \\ &= \lim_{x\to 4} \frac{(x+4)\cancel{(x-4)}}{\cancel{(x-4)}(x+3)} \\ &= \lim_{x\to 4} \frac{x+4}{x+3} \\ &= \frac{4+4}{4+3} = \frac{8}{7}. \end{aligned}$$

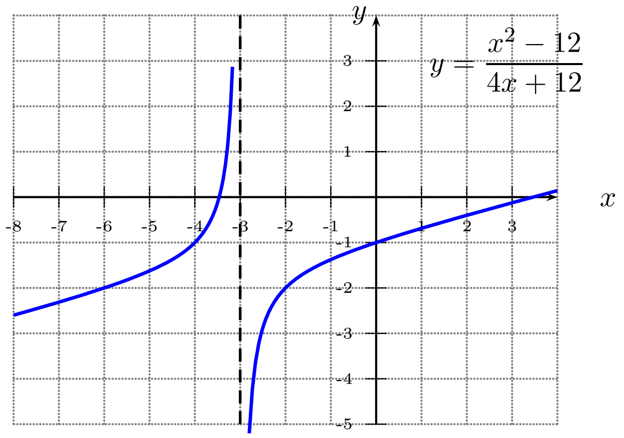
(1^{pts}) **28.** $\lim\limits_{x\to -3^-} \dfrac{x^2-12}{4x+12} =$

(a) Does Not Exist (b) ∞

(c) $-\infty$ (d) 0

Solution:

$$\begin{aligned} \lim_{x\to -3^-} \frac{x^2-12}{4x+12} &= \lim_{x\to -3^-} \frac{x^2-16}{2x+6} && \text{A direct substation gives I.F. } -7/0. \\ &= \lim_{x\to -3^-} \frac{x^2-12}{4x+12} && \text{If } x<-3 \text{ and near } -3\text{,then } x^2-12<0,\ 4x+12<0. \\ &= \lim_{x\to -3^-} \frac{\overset{-}{x^2-12}}{\overset{-}{4x+12}} = \infty \end{aligned}$$



(1pts) **29.** The curve $f(x) = \dfrac{2x^3 + x - 2}{x^3 + x - 2}$ has a horizontal asymptote at

(a) $y = 0$

(b) $x = 2$

(c) $y = 1$

(✔) $y = 2$

Solution:

To find the horizontal asymptote we take the limit as $x \to \pm\infty$ Note that both the numerator and the denominator $\to \pm\infty$, as $x \to \pm\infty$. To find this limit divided both the numerator and the denominator by the highest power of x in the denominator which is x^3. So,

$$\lim_{x\to\pm\infty} \frac{2x^3 + x - 2}{x^3 + x - 2} = \lim_{x\to\pm\infty} \frac{\dfrac{2x^3 + x - 2}{x^3}}{\dfrac{x^3 + x - 2}{x^3}}$$

$$= \lim_{x\to\pm\infty} \frac{2 + \dfrac{1}{x^2} - \dfrac{2}{x^3}}{1 + \dfrac{1}{x^2} - \dfrac{2}{x^3}}$$

$$= \frac{2 + 0 - 0}{1 + 0 - 0} = 2.$$

Therefore $y = 2$ is a horizontal asymptote.

(1pts) **30.** $\displaystyle\lim_{x\to-1} \frac{|x-1| - 2}{x + 1} =$

(a) 0

(b) Does Not Exist

(c) 1

(✔) -1

Solution:

A direct substation gives I.F. 0/0. Note that if $x > -1$ or $x < -1$ near(close) to -1

then $x-1<0 \Rightarrow |x-1| = -(x-1)$.

$$\begin{aligned}
\lim_{x\to -1} \frac{|x-1|-2}{x+1} &= \lim_{x\to 0} \frac{-(x-1)-2}{x+1} \\
&= \lim_{x\to 1} \frac{-x+1-2}{x+1} \\
&= \lim_{x\to 1} \frac{-x-1}{x+1} \\
&= \lim_{x\to 1} \frac{-(x+1)}{x+1} \\
&= \lim_{x\to 1} (-1) \\
&= -1
\end{aligned}$$