Instructions. (30 points) Solve each of the following problems.

(1pts) **1.** The period of $f(x) = \cos(2x)$ is

(✓) π (b) $\frac{\pi}{2}$

(c) 2π (d) 2

Solution:

The period of $f(x) = \cos(2x)$ is $\frac{2\pi}{2} = \pi$.

(1pts) **2.** $\cos\left(\frac{\pi}{12}\right) =$

(✓) $\frac{\sqrt{3}+1}{2\sqrt{2}}$ (b) $\frac{-\sqrt{3}+1}{2\sqrt{2}}$

(c) $-\frac{\sqrt{3}+1}{2\sqrt{2}}$ (d) $\frac{\sqrt{3}-1}{2\sqrt{2}}$

Solution:

$$\begin{aligned}
\cos\left(\frac{\pi}{12}\right) &= \cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right) && \frac{\pi}{12} = \frac{\pi}{3}-\frac{\pi}{4}, \\
&= \cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right) && \cos(A-B) = \cos A\cos B + \sin A \sin B, \\
&= \frac{1}{2}\cdot\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} && \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \sin\left(\frac{\pi}{4}\right), \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}, \cos\left(\frac{\pi}{3}\right) = \frac{1}{2} \\
&= \frac{\sqrt{3}+1}{2\sqrt{2}}
\end{aligned}$$

(1pts) **3.** If $f(x) = -x^3 - x - 1$, $x \in [0, 2]$, then the average rate of change $\frac{\Delta f}{\Delta x} =$

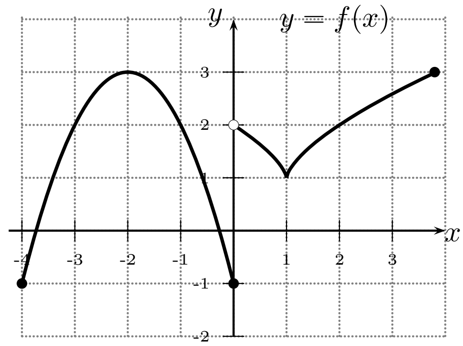
(✓) -5 (b) $\frac{1}{5}$

(c) $-\frac{1}{5}$ (d) 5

Solution:

$$\begin{aligned}
\frac{\Delta f}{\Delta x} &= \frac{f(x_2) - f(x_1)}{x_2 - x_1} \\
&= \frac{f(2) - f(0)}{2 - 0} \\
&= \frac{-11+1}{2} = -5.
\end{aligned}$$

(1pts) **4.** The accompanying figure shows the graph of $y = f(x)$. Then $f'(-2) =$.



(a) -3 (✓b) 0

(c) 1 (d) 3

Solution:

From the graph we can see that the tangent line to the graph of $y = f(x)$ at -2 is horizontal ($y = 3$) and hence its slope is zero. Now, since the first derivative is the slope of the tangent line to the graph at -2 then $f'(-2) = 0$.

(1pts) **5.** $\cos\left(x + \frac{\pi}{2}\right) =$

(a) $\cos x$ (✓b) $-\sin x$

(c) $\sin x$ (d) $-\cos x$

Solution:

$$\begin{aligned} \cos\left(x + \frac{\pi}{2}\right) &= \cos x \cos\left(\frac{\pi}{2}\right) - \sin x \sin\left(\frac{\pi}{2}\right) && \cos(A+B) = \cos A \cos B - \sin A \sin B, \\ &= \cos x \cdot 0 - \sin x \cdot 1 && \cos\left(\frac{\pi}{2}\right) = 0, \sin\left(\frac{\pi}{2}\right) = 1 \\ &= 0 - \sin x = -\sin x \end{aligned}$$

(1pts) **6.** If $y = x^2 \csc x$, then $y' =$

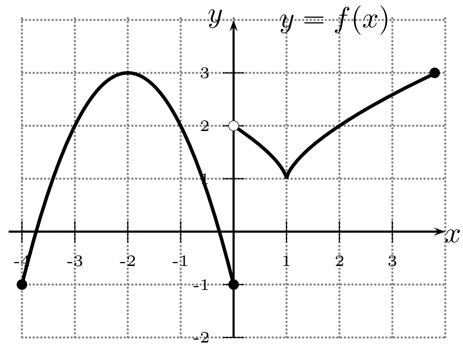
(a) $-2x \csc x \cot x + x^2 \csc x$ (✓b) $2x \csc x - x^2 \csc x \cot x$

(c) $2x \csc x + x^2 \csc x \cot x$ (d) $-2x \csc x \cot x$

Solution:

$$\begin{aligned} y &= x^2 \csc x \\ y' &= (x^2)' \csc x + x^2 (\csc x)' && \text{Use the fact } (fg)' = f'g + fg' \\ &= 2x \csc x + x^2(-\csc x \cot x) \\ &= 2x \csc x - x^2 \csc x \cot x. \end{aligned}$$

(1pts) **7.** The accompanying figure shows the graph of $y = f(x)$. Then $f'(2) > 0$.



(✓a) True

(b) False

Solution:

From the graph we can see that the tangent line to the graph of $y = f(x)$ at 2 is rising up (left to right) and hence its slope is positive. Now, since the first derivative is the slope of the tangent line to the graph at 2 then $f'(2) > 0$.

(1^{pts}) **8.** If $y = \cos x \tan x$, then $y' =$

(a) $\sec x \tan x$ (b) $-\sin x$

(✔) $\cos x$ (d) $-\csc x \cot x$

Solution:

$$\begin{aligned} y &= \cos x \tan x \\ y &= \cos x \frac{\sin x}{\cos x} \\ y &= \sin x \\ y' &= \cos x. \end{aligned}$$

(1^{pts}) **9.** If $y = 6x^2 - \dfrac{2}{x} - \dfrac{4}{x^{-1}}$, then $y'' =$

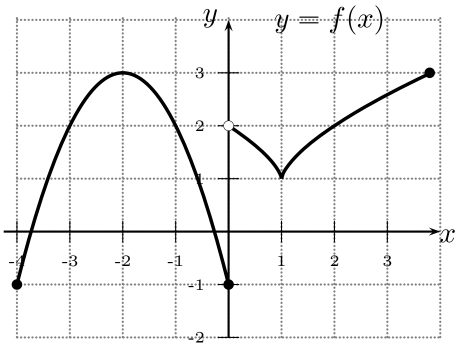
(✔) $12 - \dfrac{4}{x^3}$ (b) $12 + \dfrac{4}{x^3}$

(c) $12x + \dfrac{2}{x^2} - 4$ (d) $12 - \dfrac{24}{x^4}$

Solution:

$$\begin{aligned} y &= 6x^2 - \frac{2}{x} - \frac{4}{x^{-1}} \\ y &= 6x^2 - 2x^{-1} - 4x \\ y' &= 12x + 2x^{-2} - 4 \\ y'' &= 12 - 4x^{-3} \\ y'' &= 12 - \frac{4}{x^3}. \end{aligned}$$

(1^{pts}) **10.** The accompanying figure shows the graph of $y = f(x)$. Then f is differentiable at $x = 1$.



(a) True

(✔) False

Solution:

f is not differentiable at $x = 1$ since the derivative from the left of 1 approaches $-\infty$ and from the right of 1 approaches ∞.

(1pts) **11.** $\lim_{y\to 0} \frac{\tan(4y)}{3y} =$

(a) 3 (✔b) $\frac{4}{3}$

(c) 1 (d) $\frac{3}{4}\pi$

Solution:

Direct substitution will give us I.F. type 0/0.

$$\begin{aligned}\lim_{y\to 0} \frac{\tan(4y)}{3y} &= \frac{1}{3}\lim_{y\to 0} \frac{4\tan(4y)}{4y} \\ &= \frac{4}{3}\lim_{y\to 0} \frac{\tan(4y)}{4y} \qquad \text{use } \lim_{\theta\to 0}\frac{\tan\theta}{\theta} = 1 \\ &= \frac{4}{3}(1) = \frac{4}{3}\end{aligned}$$

(1pts) **12.** $\lim_{x\to -2} \frac{x^3+8}{x^2-4} =$

(a) −1 (b) 3

(✔c) −3 (d) 0

Solution:

$$\begin{aligned}\lim_{x\to -2} \frac{x^3+8}{x^2-4} &= \lim_{x\to -2} \frac{x^3+8}{x^2-4} && \text{A direct substitution will give us I.F. 0/0} \\ &= \lim_{x\to -2} \frac{(x+2)(x^2-2x+4)}{(x+2)(x-2)} && \text{Factoring } x+2 \text{ from denominator and numerator} \\ &= \lim_{x\to -2} \frac{\cancel{(x+2)}(x^2-2x+4)}{\cancel{(x+2)}(x-2)} \\ &= \lim_{x\to -2} \frac{x^2-2x+4}{x-2} \\ &= \frac{(-2)^2-2(-2)+4}{-2-2} = \frac{12}{-4} = -3.\end{aligned}$$

(1pts) **13.** If $y = \sqrt{2}$, then $y' =$

(a) $\frac{1}{\sqrt{2}}$ (✔b) 0

(c) $\sqrt{2}$ (d) $\frac{1}{2\sqrt{2}}$

Solution:

$y = \sqrt{2}$ the derivative of a constant is zero

$y' = 0.$

(1pts) **14.** $\lim_{x\to 0} \dfrac{x\cos(2x)}{\sin(5x)} =$

(✔) $\frac{1}{5}$ (b) $\frac{2}{5}$

(c) 5 (d) $\frac{5}{2}$

Solution:

$$\begin{aligned}
\lim_{x\to 0} \frac{x\cos(2x)}{\sin(5x)} &= \lim_{x\to 0} \frac{x}{\sin(5x)} \cdot \cos(2x) && \frac{ab}{c} = \frac{a}{b}\cdot c. \\
&= \lim_{x\to 0} \frac{5x}{5\sin(5x)} \cdot \cos(2x) && \text{make the top similar to the angle} \\
&= \frac{1}{5} \lim_{x\to 0} \frac{5x}{\sin(5x)} \lim_{x\to 0} \cos(2x) && \text{Use that } \lim_{x\to 0}\frac{\sin x}{x} = 1 = \lim_{x\to 0}\frac{x}{\sin x} \\
&= \frac{1}{5}.1.\cos 0 = \frac{1}{5}
\end{aligned}$$

(1pts) **15.** $\lim_{x\to\infty} \dfrac{2x^3+1}{6+x^2+3x^3} =$

(a) 0 (b) $\frac{3}{2}$

(✔) $\frac{2}{3}$ (d) 1

Solution:

Since $\lim_{x\to\infty}(2x^3+1) = \infty = \lim_{x\to\infty}(6+x^2+3x^3)$ we have I.F. type ∞/∞. Divide each term in the numerator and each term in the denominator by the highest power in the denominator.

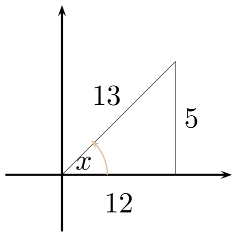
$$\begin{aligned}
\lim_{x\to\infty} \frac{2x^3+1}{6+x^2+3x^3} &= \lim_{x\to\infty} \frac{\dfrac{2x^3+1}{x^3}}{\dfrac{6+x^2+3x^3}{x^3}} \\
&= \lim_{x\to\infty} \frac{\dfrac{2x^3}{x^3} + \dfrac{1}{x^3}}{\dfrac{6}{x^3} + \dfrac{x^2}{x^3} + \dfrac{3x^3}{x^3}} \\
&= \lim_{x\to\infty} \frac{2 + \dfrac{1}{x^3}}{\dfrac{6}{x^3} + \dfrac{1}{x} + 3} \\
&= \frac{2+0}{0+0+3} = \frac{2}{3}
\end{aligned}$$

(1^{pts}) **16.** If $\tan x = \frac{5}{12}$, $0 < x < \frac{\pi}{2}$ then $\cos x =$

(a) $\frac{13}{12}$ (✔b) $\frac{12}{13}$

(c) $\frac{5}{13}$ (d) $\frac{13}{5}$

Solution:



Since $\tan x = \frac{5}{12} = \frac{\text{opposite}}{\text{adjacent}}$ we draw a triangle is shown to the right: Now, $H^2 = 5^2 + 12^2 = 25 + 144 = 169 \Rightarrow H = 13$. Since $x \in \left(0, \frac{\pi}{2}\right)$, then $\sin x \geq 0$ and $\cos x \geq 0$. Hence $\cos x = \frac{12}{13}$.

(1^{pts}) **17.** If $2 - x^2 \leq f(x) \leq 2\cos x$, then $\lim_{x\to 0} f(x) =$

(a) 0 (b) −2

(✔c) 2 (d) 1

Solution:

Since $\lim_{x\to 0}(2 - x^2) = 2 = \lim_{x\to 0} 2\cos x$, then by The Sandwich Theorem we have

$$\lim_{x\to 0} f(x) = 2.$$

(1[pts]) **18.** $\lim_{x\to 5} \frac{x^2-5x}{x^2-3x-10} =$

(a) $\frac{1}{7}$

(✔) $\frac{5}{7}$

(c) $\frac{5}{3}$

(d) 0

Solution:

$$\begin{aligned}
\lim_{x\to 5} \frac{x^2-5x}{x^2-3x-10} &= \lim_{x\to 5} \frac{x^2-5x}{x^2-3x-10} && \text{A direct substitution will give us I.F. } 0/0 \\
&= \lim_{x\to 5} \frac{x(x-5)}{(x-5)(x+2)} && \text{Factoring } x-5 \text{ from denominator and numerator} \\
&= \lim_{x\to 5} \frac{x\cancel{(x-5)}}{\cancel{(x-5)}(x+2)} \\
&= \lim_{x\to 5} \frac{x}{x+2} \\
&= \frac{5}{5+2} = \frac{5}{7}.
\end{aligned}$$

(1[pts]) **19.** $\lim_{x\to 0} \frac{|x-1|-|x+1|}{x}$

(✔) -2

(b) 0

(c) 2

(d) Does Not Exist

Solution:

A direct substitution gives I.F. 0/0. We find the limit form the right of 0 and the left of 0.

$$\begin{aligned}
\lim_{x\to 0^+} \frac{|x-1|-|x+1|}{x} &= \lim_{x\to 0^+} \frac{|x-1|-|x+1|}{x} && x\to 0^+ \Rightarrow x>0,\ x>-1,\ x<1, \\
&= \lim_{x\to 0^+} \frac{-(x-1)-(x+1)}{x} && \text{hence } |x-1| = -(x-1),\ |x+1| = x+1 \\
&= \lim_{x\to 0^+} \frac{-x+1-x-1}{x} \\
&= \lim_{x\to 0^+} \frac{-2\cancel{x}}{\cancel{x}} \\
&= \lim_{x\to 0^+} -2 = -2
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to 0^-} \frac{|x-1|-|x+1|}{x} &= \lim_{x\to 0^-} \frac{|x-1|-|x+1|}{x} && x\to 0^- \Rightarrow x<0, x>-1,\ x<1, \\
&= \lim_{x\to 0^-} \frac{-(x-1)-(x+1)}{x} && \text{hence } |x-1| = -(x-1),\ |x+1| = x+1 \\
&= \lim_{x\to 0^-} \frac{-x+1-x-1}{x} \\
&= \lim_{x\to 0^-} \frac{-2\cancel{x}}{\cancel{x}} \\
&= \lim_{x\to 0^-} -2 = -2
\end{aligned}$$

Now, since $\lim_{x\to 0^+} \frac{|x-1|-|x+1|}{x} = -2 = \lim_{x\to 0^-} \frac{|x-1|-|x+1|}{x}$, then

$$\lim_{x\to 0} \frac{|x-1|-|x+1|}{x} = -2.$$

(1^{pts}) **20.** $\lim_{x\to 2} \frac{\sqrt{x+2}-2}{x-2} =$

(a) 0

(b) $-\frac{1}{4}$

(✔) $\frac{1}{4}$

(d) 4

Solution:

A direct substitution will give us I.F. 0/0. Now,

$$\begin{aligned}
\lim_{x\to 2} \frac{\sqrt{x+2}-2}{x-2} &= \lim_{x\to 2} \frac{\sqrt{x+2}-2}{x-2} \cdot \frac{\sqrt{x+2}+2}{\sqrt{x+2}+2} && \text{Multiply by } 1 = \frac{\sqrt{x+2}+2}{\sqrt{x+2}+2} \\
&= \lim_{x\to 2} \frac{(\sqrt{x+2})^2-(2)^2}{(x-2)(\sqrt{x+2}+2)} \\
&= \lim_{x\to 2} \frac{x+2-4}{(x-2)(\sqrt{x+2}+2)} && \text{Simplify} \\
&= \lim_{x\to 2} \frac{\cancel{(x-2)}}{\cancel{(x-2)}(\sqrt{x+2}+2)} \\
&= \lim_{x\to 2} \frac{1}{\sqrt{x+2}+2} \\
&= \frac{1}{\sqrt{2+2}+2} \\
&= \frac{1}{4}.
\end{aligned}$$

(1^{pts}) **21.** $\lim_{x\to 2^+} \frac{x-1}{x-2} =$

(a) $-\infty$

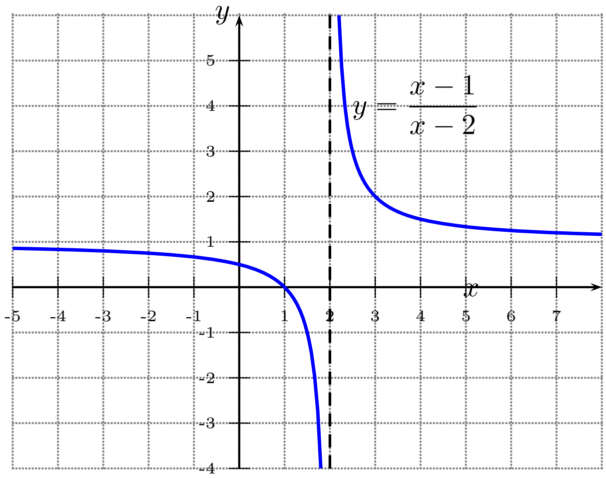
(b) 1

(c) 0

(✔) ∞

Solution:

$$\begin{aligned}
\lim_{x\to 2^+} \frac{x-1}{x-2} &= \lim_{x\to 2^+} \frac{x-1}{x-2} && \text{A direct substitution gives I.F. } 3/0. \\
&= \lim_{x\to 2^+} \frac{x-1}{x-2} && \text{If } x>2 \text{ and near 2,then } x-1>0,\ x-2>0. \\
&= \lim_{x\to 2^+} \frac{\overset{+}{x-1}}{\underset{x-2}{+}} = \infty
\end{aligned}$$


y
x
$y = \frac{x-1}{x-2}$
5
4
3
2
1
-1
-2
-3
-4
-5
-4
-3
-2
-1
1
2
3
4
5
6
7

(1pts) **22.** An equation for the tangent line to $f(x) = \dfrac{2}{x^2+1}$ at the point $(1,1)$ is

(✔) $y = -x + 2$ (b) $y = x$

(c) $y = 2x - 1$ (d) $y = -2x + 3$

Solution:

The slope of the tangent line to $f(x) = \dfrac{2}{x^2+1}$ at $x = 1$ is $f'(1)$.

$$f(x) = \frac{2}{x^2+1} \Rightarrow f'(x) = \frac{(x^2+1)(0) - 2(2x)}{(x^2+1)^2} = \frac{-4x}{(x^2+1)^2}.$$

Hence

$$\text{the slope of the tangent } = f'(1) = \frac{-4(1)}{((1)^2+1)^2} = -1.$$

Now, we have $m = -1$ and $(1,1)$, hence

$$y - y_1 = m(x - x_1) \Rightarrow y - 1 = -1(x - 1) \Rightarrow y - 1 = -x + 1 \Rightarrow y = -x + 2.$$

(1pts) **23.** The curve $f(x) = \dfrac{2x^2 - x + 1}{x^2 - 4x + 1}$ has horizontal asymptote at

(a) $x = 2$ (✔) $y = 2$

(c) $y = 0$ (d) $y = \pm 2$

Solution:

To find the horizontal asymptote we take the limit as $x \to \pm\infty$. Note that both the numerator and the denominator $\to \pm\infty$, as $x \to \pm\infty$. To find this limit divided both the numerator and the denominator by the highest power of x in the denominator which is x^2. So,

$$\lim_{x\to\pm\infty} \frac{2x^2 - x + 1}{x^2 - 4x + 1} = \lim_{x\to\pm\infty} \frac{\dfrac{2x^2 - x + 1}{x^2}}{\dfrac{x^2 - 4x + 1}{x^2}}$$

$$= \lim_{x\to\pm\infty} \frac{2 - \dfrac{1}{x} + \dfrac{1}{x^2}}{1 - \dfrac{4}{x} + \dfrac{1}{x^2}}$$

$$= \frac{2 - 0 + 0}{1 - 0 + 0} = 2.$$

Therefore $y = 2$ is a horizontal asymptote.

(1pts) **24.** If $y = \dfrac{x-1}{x+1}$, then $y' =$

(✔) $\dfrac{2}{(x+1)^2}$ (b) $\dfrac{2x}{(x+1)^2}$

(c) $\dfrac{-2x}{(x+1)^2}$ (d) $\dfrac{-2}{(x+1)^2}$

Solution:

$$y' = \frac{\overbrace{(x+1)}^{\text{Bottom}}\ \overbrace{(x-1)'}^{\text{Derivative of Top}} - \overbrace{(x-1)}^{\text{Top}}\ \overbrace{(x+1)'}^{\text{Derivative of Bottom}}}{\underbrace{}\overbrace{(x+1)^2}^{\text{Bottom}^2}}$$

$$= \frac{(x+1)(1)-(x-1)(1)}{(x+1)^2}$$

$$= \frac{x+1-(x-1)}{(x+1)^2}$$

$$= \frac{x+1-x+1}{(x+1)^2}$$

$$= \frac{2}{(x+1)^2}.$$

(1^{pts}) **25.** The function $f(x) = \dfrac{x+1}{x^2-4x+3}$ is discontinuous at

(a) $x=-1$, $x=3$ (b) $x=1$, $x=-3$

(c) $x=-1$, $x=-3$ (✔d) $x=1$, $x=3$

Solution: Notice that $f(x) = \dfrac{x+1}{x^2-4x+3}$ is a rational function, then it is continuous on its domain which is the set of real number except the zeroes of the denominator and it is discontinuous at the zeroes of the the denominator. Now

$$x^2-4x+3=0 \Leftrightarrow (x-1)(x-3)=0$$
$$\Leftrightarrow x=1 \text{ or } x=3.$$

Hence f is discontinuous at $x=1$, $x=3$.

(1^{pts}) **26.** If $y = \dfrac{x^2+x+1}{x}$, then $y' =$

(a) $1 + \dfrac{1}{x}$

(b) $1 - \dfrac{1}{x}$

(✔) $1 - \dfrac{1}{x^2}$

(d) $1 + \dfrac{1}{x^2}$

Solution:

$$\begin{aligned} y &= \frac{x^2+x+1}{x} \\ y &= x + 1 + x^{-1} \\ y' &= 1 - x^{-2} \\ y' &= 1 - \frac{1}{x^2}. \end{aligned}$$

(1^{pts}) **27.** If $y = \dfrac{\sin x}{1+\sin x}$, then $y' =$

(a) 1

(b) $\dfrac{-\cos x}{(1+\sin x)^2}$

(✔) $\dfrac{\cos x}{(1+\sin x)^2}$

(d) $\dfrac{\sin x}{(1+\sin x)^2}$

Solution:

$$\begin{aligned} y' &= \frac{(1+\sin x)(\cos x) - (\sin x)(\cos x)}{(1+\sin x)^2} \\ &= \frac{\cos x + \sin x\cos x - \sin x \cos x}{(1+\sin x)^2} \\ &= \frac{\cos x}{(1+\sin x)^2}. \end{aligned}$$

(1^{pts}) **28.** The curve $f(x) = \dfrac{x+3}{x^2+2x-3}$, has a vertical asymptote at

(✔) $x = 1$

(b) $x = 1, \quad x = -3$

(c) $y = 1$

(d) $y = 1, \quad y = -3$

Solution:

Write $f(x) = \dfrac{x+3}{(x-1)(x+3)}$. The zeroes of the denominator are $1, -3$.
To chuck that $x = -3$ is a vertical asymptote or not we take the limit at -3 from both sides.

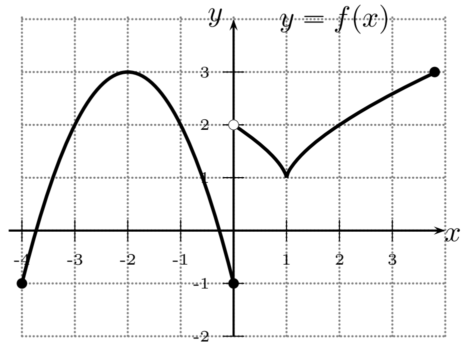
$$\lim_{x\to -3} f(x) = \lim_{x\to -3} \frac{\cancel{(x+3)}}{\cancel{(x+3)}(x-1)} = \lim_{x\to -3} \frac{1}{x-1} = -\frac{1}{4}.$$

Hence $x = -3$ is not a vertical asymptote.
To chuck that $x = 1$ we take the limit

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{\overset{+}{x+3}}{\overset{+}{(x-1)}(x+3)} = \infty.$$

Hence $x = 1$ is a vertical asymptote.

(1pts) **29.** The accompanying figure shows the graph of $y = f(x)$. Then f is continuous at $x = 0$.



(a) True

(✔b) False

Solution:
since $\lim_{x\to 0^+} f(x) = 2 \neq -1 = \lim_{x\to 0^-} f(x)$, then $\lim_{x\to 0} f(x)$ does not exist.

(1pts) **30.** $\lim_{x\to -3} \frac{x^2 - 2x - 15}{x^2 + 9} =$

(✔a) 0

(b) $\frac{4}{3}$

(c) $\frac{-4}{3}$

(d) Does Not Exist

Solution:

$$\begin{aligned}\lim_{x\to -3} \frac{x^2 - 2x - 15}{x^2 + 9} &= \frac{(-3)^2 - 2(-3) - 15}{(-3)^2 + 9} \\ &= \frac{9 + 6 - 15}{9 + 9} \\ &= \frac{0}{18} = 0\end{aligned}$$